Arithmetic and Geometric Series: Review

Fact — We can write sequences in different ways:

1. $x_n = g(n)$ - using a formula
2. $x_{n+1} = f(x_n), x_1 = a$ - using a rule based on previous terms.

Tip

Don't forget the first term if using a recursive formula!

Example

Write down using both forms, formulae for:

- $1, 5, 9, 13, \ldots$
- $12, 6, 3, \frac{3}{2}, \ldots$

- *This is an* arithmetic sequence, *first term* 1, *common difference* 4
  1. $x_n = 1 + 4(n-1) = 4n - 3$
  2. $x_{n+1} = x_n + 4, \quad x_1 = 1$
- *This is a* geometric sequence, *first term* 12, *common ratio* $\frac{1}{2}$
  1. $x_n = 12 \cdot \frac{1}{2^{n-1}}$
  2. $x_{n+1} = \frac{1}{2} x_n, \quad x_1 = 12$

Fact — For an arithemtic sequence, with first term a and common difference d, the sum of the first n terms is:

$$S_n = \underbrace{n}_{\text{number of terms}} \left(\underbrace{\frac{a + a + (n-1)d}{2}}_{\text{average of first and last terms}} \right) = \frac{n}{2}(2a + (n-1)d)$$

For a geometric sequence, with first term a and common ratio r, the sum of the first n terms is:

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

If $|r| < 1$, then

$$S_\infty = \frac{a}{1-r}$$

Example

A student is reading a 426-page book finds that he reads faster as he gets into the subject. He reads 19 pages on the first day, and his rate of reading then goes up by 3 pages each day. How long does he take to finish the book?

$a = 19, d = 3, S = 426$, so

$$\begin{aligned}
& 426 = \frac{n}{2}(38 + (n-1)3) \\
\Rightarrow \quad & 852 = n(3n+35) \\
\Rightarrow \quad & 0 = 3n^2 + 35n - 852 \\
\Rightarrow \quad & n = \frac{-35 \pm \sqrt{35^2 - 4 \cdot 3 \cdot (-852)}}{2 \cdot 3} \\
& = \frac{-35 \pm 107}{6} \\
\Rightarrow \quad & n = \frac{-35 + 107}{6} = \frac{72}{6} = 12
\end{aligned}$$

He will finish the book in 12 days.

Tip

If a, b, c are in arithmetic progression, then:

$$b - a = c - b \Leftrightarrow b = \frac{a + c}{2}$$

If a, b, c are in geometric progression, then:

$$\frac{b}{a} = \frac{c}{b} \Leftrightarrow b^2 = ac$$

Solving equations numerically: Review

Fact — If the sequence given by the recursive definition $x_{r+1} = F(x_r)$, with some initial value *converges to a limit* l, then l is a root of the equation $x = F(x)$

Tip

A sequence given be a recursive definition $x_{r+1} = F(x_r)$ doesn't necessarily converge!

Example

Suppose we want to solve the equation $x^3 - 3x - 5 = 0$. We can write this as:

- $x = \frac{1}{3}(x^3 - 5)$, ie $x_{r+1} = \frac{1}{3}(x_r^3 - 5)$
- $x = \sqrt[3]{3x+5}$, ie $x_{r+1} = \sqrt[3]{3x_r + 5}$

Starting from $x_0 = 2$, write down first 5 terms of each iteration. Find a root to 3 (s.f.).

r	x_r
0	2
1	1
2	−1.33333
3	−2.45679
4	−6.60958
5	−97.91654

r	x_r
0	2
1	2.22398
2	2.26837
3	2.27697
4	2.27862
5	2.27894

A plausible guess for a root to 3 (s.f.) would be 2.28, but we must check this exactly, so, we use the sign change rule.

$2.275 - F(2.275) = -3.24 \times 10^{-3} < 0$, $2.28 - F(2.28) = 4.83 \times 10^{-3} > 0$, *therefore since the sign changes over the interval* $[2.275, 2.285]$ *and all values in that interval are* 2.28 *to* 3 *(s.f.) this must be a root to* 3 *(s.f.)*

Fact (Convergence of Iterative Methods) — If the iteration $x_{r+1} = F(x_r)$ is used to find approximations of a root α, then the sequence of errors is approximately *geometric* with common ratio $F'(\alpha)$ (assuming $F'(\alpha) \neq 0$)

Fact (Convergence of Iterative Methods) — If the method has $F'(\alpha) = 0$ then the convergence is *quadratic* (or better).

Fact (Newton-Raphson) — To find a root of $f(x) = 0$, consider using the iterative method:

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

ie with $F(x) = x - \dfrac{f(x)}{f'(x)}$

This is useful, since $l = F(l) \Rightarrow l - f(l)/f'(l) = l \Rightarrow f(l) = 0$, so if the recurrence converges, it converges to a root of $f(x)$.

It's doubly useful since

$$\begin{aligned} F'(x) &= 1 - \frac{f'(x)^2 - f(x)f''(x)}{(f'(x))^2} \\ &= \frac{f'(x)^2 - f'(x)^2 + f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \end{aligned}$$

So $F'(l) = 0$ and we have quadratic convergence

Example

Suppose $f(x) = x^3 - 3x - 5$, use Newton-Raphson to find a **root** near 2 to 6 (d.p.)

$F(x) = x - \frac{x^3-3x-5}{3x^2-3}$ and so

r	x_r
0	2
1	2.33333
2	2.28055
3	2.279020068
4	2.279018786
5	2.279018786

A quick check of $f(2.2790185)$ and $f(2.2790195)$ shows that 2.279019 is a root to 6 (d.p)

Standard Series

Fact —

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Tip

You need to be able to prove these results! (Induction!)

Example

Prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ by induction

Proof: (By induction)
(Base Case): $n = 1$

$$LHS = \sum_{r=1}^{1} r^3 = 1$$
$$RHS = \frac{1^2(1+1)^2}{4} = 1$$

So the result is true for $n = 1$.
(Inductive Step): *Suppose the result is true for* $n = k$:

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

Consider $n = k+1$, *then*

$$\begin{aligned}\sum_{r=1}^{k+1} r^3 &= \sum_{r=1}^{k} r^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= (k+1)^2\left(\frac{1}{4}k^2 + (k+1)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{(k+1)^2}{4}(k^2+4k+4) \\
&= \frac{(k+1)^2}{4}(k+2)^2 \\
&= \frac{1}{4}(k+1)^2(k+2)^2
\end{aligned}
$$

So the result is also true for $n = k+1$.

(Conclusion): *Since the result is true for $n = 1$, and if the result is true for $n = k$ it is true for $n = k+1$, by the principle of mathematical induction it is true for all $n \geq 1$*

Fact —

$$
\begin{aligned}
\sum (a_n + b_n) &= \sum a_n + \sum b_n \\
\sum c a_n \qquad &= c \sum a_n
\end{aligned}
$$

Example

Find a formula for $1 \times 2 \times 4 + 2 \times 3 \times 5 + \cdots + n(n+1)(n+3)$.

The r^{th} term is $r(r+1)(r+3)$, so we are computing:

$$
\begin{aligned}
\sum_{r=1}^{n} r(r+1)(r+3) &= \sum_{r=1}^{n} (r^3 + 4r^2 + 3r) \\
&= \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} 4r^2 + \sum_{r=1}^{n} 3r \\
&= \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r \\
&= \frac{n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} \\
&= \frac{1}{12} n(n+1)(3n(n+1) + 8(2n+1) + 18) \\
&= \frac{1}{12} n(n+1)\left(3n^2 + 19n + 26\right)
\end{aligned}
$$

Method of Differences - Telescoping Series

Example

Find the sum $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2+k}$

$$
\begin{aligned}
& & \frac{1}{k^2+k} &= \frac{1}{k(k+1)} \\
& & &= \frac{A}{k} + \frac{B}{k+1} \\
&\Rightarrow & 1 &= A(k+1) + Bk \\
k = 0 :& & A &= 1 \\
k = -1 :& & B &= -1 \\
&\Rightarrow & \frac{1}{k^2+k} &= \frac{1}{k} - \frac{1}{k+1}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) &= \left(\frac{1}{1} - \frac{1}{1+1}\right) + \left(\frac{1}{2} - \frac{1}{2+1}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \\
&= \frac{n}{n+1}
\end{aligned}
$$

Fact (Method of Differences) — If $g(r) = f(r+1) - f(r)$

$$\sum_{r=1}^{n} g(r) = \sum_{r=1}^{n} \left(f(r+1) - f(r)\right) = f(n+1) - f(1)$$

Example

Expand $(r+1)^3 - r^3$, and use your result to derive a formula for $\sum_{r=1}^{n} r^2$.

$$\begin{aligned}
(r+1)^3 - r^3 &= (r^3 + 3r^2 + 3r + 1) - r^3 \\
&= 3r^2 + 3r + 1 \\
\Rightarrow \qquad \sum_{r=1}^{n}(3r^2 + 3r + 1) &= \sum_{r=1}^{n}\big((r+1)^3 - r^3\big) \\
&= (n+1)^3 - 1^3 \\
&= (n+1)^3 - 1
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{n}(3r^2 + 3r + 1) &= 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \\
&= 3\sum_{r=1}^{n} r^2 + 3\frac{n(n+1)}{2} + n \\
\Rightarrow \qquad (n+1)^3 - 1 &= 3\sum_{r=1}^{n} r^2 + 3\frac{n(n+1)}{2} + n \\
\Rightarrow \qquad 3\sum_{r=1}^{n} r^2 &= (n+1)^3 - 1 - \frac{3}{2}n(n+1) - n \\
&= (n+1)^3 - \frac{3}{2}n(n+1) - (n+1) \\
&= (n+1)((n+1)^2 - \frac{3}{2}n - 1) \\
&= (n+1)(n^2 + 2n + 1 - \frac{3}{2}n - 1) \\
&= (n+1)(n^2 + \frac{1}{2}n) \\
&= \frac{1}{2}n(n+1)(2n+1) \\
\Rightarrow \qquad \sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1)
\end{aligned}$$

Proof by Induction

Example

Prove that $3 \times 7^{2n} + 1$ is divisible by 4 for $n \in \mathbb{N}$

Proof: (By Induction)
Base Case: $n = 1$
$3 \times 7^2 + 1 = 3 \times 49 + 1 = 147 + 1 = 148 = 4 \times 37$ *which is clearly divisible by* 4.
Inductive step: Suppose our result is true for some $n = k$, *ie* $3 \times 7^{2k} + 1 = 4m$, *the consider* $n = k + 1$,

$$\begin{aligned}
3 \times 7^{2(k+1)} + 1 &= 3 \times 7^{2k} \times 7^2 + 1 \\
&= (\underbrace{3 \times 7^{2k} + 1}_{\text{our expression}} - 1) \times 7^2 + 1 \\
&= (4m - 1) \times 49 + 1 \\
&= 4m \times 49 - 49 + 1 \\
&= 4m \times 49 - 48 \\
&= 4(49m - 12)
\end{aligned}$$

which is divisible by 4.

Conclusion: Notice that our statement is true for $n = 1$ *and if our statement is true fro* $n = k$, *then it's also true for* $n = k + 1$, *but this means by the principle of mathematical induction it is true for all integer* $n \geq 1$, *ie* $n \in \mathbb{N}$

Example

Prove that $2^n > 2n$ for $n \geq 3$

Proof: (By induction)
Base case: $n = 3$ [Notice not $n = 1$!].
$LHS = 2^3 = 8 > 6 = 2 \cdot 3 = RHS$, therefore our statement is true for $n = 3$.
Inductive step: Assume our statement is true for some $n = k$, ie $2^k > 2k$, then consider $n = k + 1$,

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot 2k && \text{(assumption)} \\ &= 4k \\ &= \underbrace{2k + 2}_{\textit{target}} + \underbrace{2k - 2}_{\textit{leftovers}} \\ &= 2(k + 1) + 2(k - 1) \\ &> 2(k + 1) && \text{(since } k - 1 > 0\text{)} \end{aligned}$$

Therefore if our statement is true for $n = k$, it is true for $n = k + 1$.
Since our statement is true for $n = 3$ and if it is true for $n = k$, it is true for $n = k + 1$, by the principle of mathematical induction it is true for all integer $n \geq 3$.

Example

Use induction to prove that the inequality $2^n > 6n + 1$ holds for all integers $n \geq 5$.

Some answer

Example

Suppose $x_{n+1} = 3x_n - 1, x_1 = 2$, then $x_n = \frac{1}{2}(3^n + 1)$

Proof: (By Induction)
Base case: $n = 1$*:* $x_1 = \frac{1}{2}(3+1) = 2$ ✓
Inductive step: Suppose our statement is true for $n = k$*, ie* $x_k = \frac{1}{2}(3^k + 1)$*, then consider* $n = k + 1$

$$\begin{aligned} x_{k+1} &= 3x_k - 1 \\ &= \frac{3}{2}(3^k + 1) - 1 \\ &= \frac{3^{k+1} + 3 - 2}{2} \\ &= \frac{3^{k+1} + 1}{2} \end{aligned}$$

Therefore our statement is true for $n = k + 1$ *Conclusion: Since our statement is true for* $n = 1$*, and if it is true for* $n = k$*, it is true for* $n = k + 1$*, we can conclude from the principle of mathematical induction it is true for all* $n \geq 1$

Example

A sequence of integers is defined recursively by the relation

$$a_{n+1} = a_n - 4, a_1 = 3, n = 1, 2, 3, \ldots$$

Prove by induction that its n^{th} term is given by

$$a_n = 7 - 4n$$

Proof: (By induction)
Base case: If $n = 1$*,* $7 - 4 \cdot 1 = 3 = a_1$*, therefore our statement is true.*
Inductive step: Suppose our statement is true for $n = k$*, ie* $a_k = 7 - 4k$*, the consider* $n = k + 1$*,*

$$\begin{aligned} a_{k+1} &= a_k - 4 \\ &= 7 - 4k - 4 \\ &= 7 - 4(k + 1) \end{aligned}$$

Therefore our statement is true for $n = k + 1$ *Conclusion: Since our statement is true for* $n = 1$*, and if it is true for* $n = k$*, it is true for* $n = k + 1$*, we can conclude from the principle of mathematical induction it is true for all* $n \geq 1$